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BOUNDARY LAYER IN THERMAL RADIATION ABSORBING AND EMITTING MEDIA†

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Abstract—A boundary-layer problem concerned with the effects of thermal radiation on the temperature distribution and the heat transfer in an absorbing and emitting media flowing over wedge is considered. The general boundary-layer problem is formulated in terms of an equation of transfer for thermal radiation, a nonlinear integrodifferential equation which represents conservation of energy, and the usual equations for conservation of mass and momentum. Because no general solution for these equations is possible, the Rosseland approximation for the radiant heat flux vector is used to simplfy the energy equation. Numerical solutions for temperature distribution and heat transfer are given. Calculations are presented for a fluid whose Prandtl number is 1, as well as for molten Pyrex glass.

NOMENCLATURE

c, velocity of light;
c_p, specific heat at constant pressure;
E_{bb}, black-body emissive power, $E_{bb} = \pi I_{bb} = \sigma T^4$;
$\mathbf{E}$, radiant energy flux vector defined by equation (5);
$\mathscr{E}$, incident radiation defined by equation (4);
f, dimensionless stream function defined by equation (17);
h, enthalpy;
I, intensity of radiation or the amount of radiant energy in a pencil of rays per unit time, solid angle, and the area perpendicular to the ray;
I_{bb}, Planck's function of black-body radiation;
k, thermal conductivity;
k_{eff}, effective thermal conductivity defined by equation (13);
m, exponent in equation (15);
N_{Pr}, Prandtl number;
N, dimensionless parameter, $N = \kappa/4n^2\sigma T^{*3}$;
n, index of refraction;
p, pressure;
$\mathbf{r}$, position vector;
s, position co-ordinate in the direction of unit vector $\mathbf{\Omega}$;
T, absolute temperature;
t, time;
U, velocity outside the boundary layer;
u, radiant-energy density defined by equation (4);
u, velocity in the x-direction;
u_1, constant defined by equation (17);
v, velocity in the y-direction;
x, position co-ordinate;
β, pressure gradient parametered, $\beta = 2m/(m + 1)$;
γ, extinction coefficient or sum of absorption and scattering coefficients;
η, similarity variable defined by equation (16);
θ, dimensionless (normalized) temperature, $\theta = T/T^*$;
κ, absorption coefficient;
λ, wavelength;
μ, dynamic viscosity;
ν, kinematic viscosity;
ρ, density;
σ, Stefan–Boltzmann constant;
Φ, disspation function;
ψ, stream function defined by equation (17);

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- Ω, solid angle;
- $\mathbf{\Omega}$, unit vector;
- ∇, vector operator used for, say, the gradient of a scalar.

Subscripts

- 0, free stream;
- w, wall;
- λ, monochromatic (a given wavelength or per unit wavelength).

Superscripts

- * property evaluated at an arbitrary temperature T^*;
- ′ denotes differentiation with respect to the similarity variable η.

INTRODUCTION

TWO-DIMENSIONAL laminar boundary layer-flow and convective heat transfer have been studied by many investigators; not much attention has been given, however, to cases where thermal radiation becomes an additional factor. Recent developments in hypersonic flight, missile re-entry, rocket combustion chambers, power plants for interplanetary flight and gas-cooled nuclear reactors have focused attention on thermal radiation as a mode of energy transfer, and emphasized the need for an improved understanding of radiative transfer in these processes.

Problems where convection and radiation are coupled have been receiving more attention [1–7]. In these studies, simplifying assumptions and approximations were introduced to obtain the solution of the conservation equations or of only the energy equation, and only Howe [4] and Kadanoff [7] presented numerical results. In his analysis, Howe assumed that the opaque gas only absorbs, but does not emit any thermal radiation. Kadanoff used the integrated (over-all solid angles) form of the equation of transfer, otherwise known as the Milne–Eddington approximation. Ruminskiĭ [5] employed a similar equation to express the conservation of radiant energy. The Milne–Eddington approximation, known as the first-order diffusion approximation, was developed to deal with the problems in astrophysics and neutron transport [8].

These studies have shown that thermal radiation affects heat transfer both directly and indirectly. Radiant energy can be directly absorbed or emitted by a surface and thereby alter its heat-transfer characteristics. Indirectly, thermal radiation can be partially absorbed in the boundary layer, altering the temperature distribution, and thereby influencing the conductive and convective heat-transfer characteristics.

The purpose of this paper is to study the effects of thermal radiation on the temperature distribution and heat transfer during flow of an absorbing and emitting medium along a wedge so that a better understanding of combined radiation and convection might be possible. Since the complexity introduced by radiative contribution to the energy flux is in part due to the dependence of the flux on the geometrical configuration of the problem, the simple geometry of a wedge was chosen.

When energy transfer by radiation and convection is present, the law of conservation of energy becomes a complicated nonlinear integro-differential equation. Thus, an exact analysis of the interaction of a radiation field with absorbing and emitting media in the laminar boundary layer is a very difficult problem. For this reason, the Rosseland approximation, which is valid for intense absorption, is used below to approximate the radiant-energy flux vector.

The strongly absorbing medium is of interest as it provides a physically significant standard of comparison for understanding the general case, and the authors are of the opinion that the simple problem must be solved first before the more difficult problem can be attempted.

BASIC CONSERVATION EQUATIONS

The first step in the analysis of a combined convection and radiation heat-transfer problem is the application of conservation laws of mass, momentum and energy. In nonrelativistic terms, the presence of radiation does not affect the equations of mass and momentum, since radiation has no mass and the "radiative viscosity" is very small at ordinary temperatures. The classical forms of these equations are well known and therefore are not considered here. Attention is focused below on the equation of transfer, or the equation of monochromatic radiant-energy conservation, and on the equation of energy including the terms due to thermal radiation.

Equation of transfer

Before the equation for conservation of energy, which customarily accounts for conduction, convection and diffusion and usually omits radiation as a mode of energy transfer, can be derived, the equation of transfer of thermal radiation in a radiating medium must be considered.

The equation of transfer can be derived by considering a small cylindrical element, Fig. 1, of cross-section dA and length ds in an absorbing and emitting medium in which temperature varies from point to point in space and time. Radiant energy in the wavelength interval between λ and $\lambda + d\lambda$ and confined to a pencil of rays of solid angle dΩ about the direction of the unit vector $\mathbf{\Omega}$ will cross the two faces normally in a time interval dt. As measured by the monochromatic intensity of radiation, $I_\lambda(\mathbf{r}, \mathbf{\Omega}, t)$, the intensity will decrease on account of absorption (scattering is neglected) and increase because of contributions from the emission in the volume element. Counting up the gains and losses of radiant energy in the pencil of rays dΩ during its traversal of distance ds, one can show [9]

$$\frac{\partial I_\lambda(\mathbf{r}, \mathbf{\Omega}, t)}{c\partial t} + \nabla \times [\mathbf{\Omega} I_\lambda(\mathbf{r}, \mathbf{\Omega}, t)] = \kappa_\lambda[n_\lambda^2 I_{bb,\lambda}(\mathbf{r}, t) - I_\lambda(\mathbf{r}, \mathbf{\Omega}, t)], \quad (1)$$

where $I_{bb,\lambda}$ is Planck's function. The first and second terms on the left-hand side of this equation give the change in monochromatic intensity with time and position, respectively. The first and second terms on the right-hand side denote emission and absorption of the radiant energy.

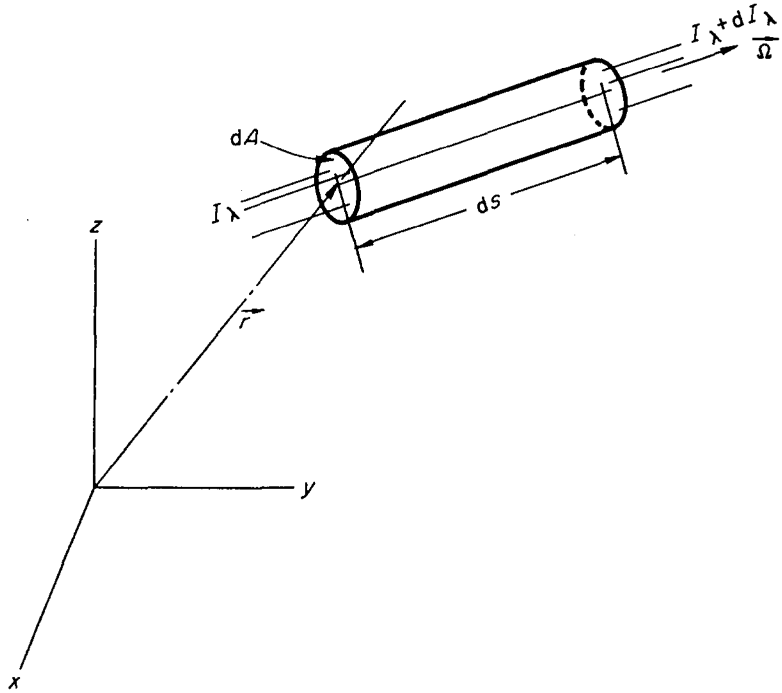

FIG. 1. Co-ordinates for equation of transfer.

The formal solution of equation (1) is readily written down. By considering a pencil of rays in the direction $\mathbf{\Omega}$, the directional derivative becomes $\nabla \times (\mathbf{\Omega} I_\lambda) = dI_\lambda/ds$, and the steady-state solution of equation (1) becomes

$$I_\lambda(s) = I_\lambda(0) \exp\left(-\int_0^s \kappa_\lambda \, ds\right) + \int_0^s n_\lambda^2 \kappa_\lambda I_{bb,\lambda}(s') \exp\left(-\int_{s'}^s \kappa_\lambda \, ds\right) ds', \quad (2)$$

where $I_\lambda(0)$ is the monochromatic intensity of radiation in the direction $\mathbf{\Omega}$ leaving the surface at $s = 0$. It should be noted that equation (2) does not in any real sense solve the equation of transfer because the temperature distribution ($I_{bb,\lambda}$) must still be specified to determine the intensity of radiation.

Integrating equation (1) over all solid angles ($\Omega = 4\pi$), there is obtained [9]

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = -\nabla \times \mathbf{E}_\lambda(\mathbf{r}, t) + \kappa_\lambda[4n_\lambda^2 E_{bb,\lambda}(\mathbf{r}, t) - \mathscr{E}_\lambda(\mathbf{r}, t)], \quad (3)$$

where the radiant-energy density $u_\lambda(\mathbf{r}, t)$ is defined as

$$u_\lambda(\mathbf{r}, t) = \mathscr{E}_\lambda(\mathbf{r}, t)/c = 1/c \int_{\Omega=4\pi} I_\lambda(\mathbf{r}, \mathbf{\Omega}, t)\, d\Omega \quad (4)$$

and the monochromatic radiant-energy transfer per unit cross-sectional area perpendicular to a unit vector $\mathbf{\Omega}$ is defined as

$$\mathbf{E}_\lambda(\mathbf{r}, t) = \int_{\Omega=4\pi} I_\lambda(\mathbf{r}, \mathbf{\Omega}, t)\mathbf{\Omega}\, d\Omega. \quad (5)$$

Equation (3) describes the local rate of change of monochromatic radiant energy density.

The equation for conservation of total† radiant energy is obtained, of course, by integrating equation (3) over all wavelengths from 0 to ∞. This introduces additional difficulties because of the double integrals involved. Considerable simplification is obtained, however, for the case in which the monochromatic absorption coefficient is independent of wavelength or nearly so, in that κ_λ can be replaced by an appropriate mean absorption coefficient κ. The index of refraction is ordinarily quite uniform over a range of wavelengths.

There is lack of agreement [8] on the most appropriate definition for the mean absorption coefficient. However, Planck's definition

$$\kappa \equiv \int_0^\infty \kappa_\lambda I_{bb,\lambda}\, d\lambda \Big/ \int_0^\infty I_{bb,\lambda}\, d\lambda \quad (6)$$

seems to be the most meaningful when energy transport by conduction and convection are present and therefore is used in this paper. Very little data are available on the dependence of the radiative properties on wavelength, temperature and pressure; they have been investigated only in certain wavelength regions and only for low pressures and temperatures. Since so little information is available, numerical calculations which follow will treat the absorption coefficient as being independent of position and wavelength.

Energy equation

With energy transfer by conduction, convection and radiation and work, the equation for conservation of energy can be written as [9]

$$\frac{\partial u}{\partial t} + \rho \frac{Dh}{Dt} = \nabla \times (k\nabla T - \mathbf{E}) + \frac{Dp}{Dt} + \mu\Phi. \quad (7)$$

The first term on the left-hand side and the term $\nabla \cdot \mathbf{E}$ on the right-hand side represent the net rate of gain of energy per unit of volume due to thermal radiation. These two terms are ordinarily neglected in standard works on convective heat transfer.

Integrating equation (3) over all wavelengths and substituting in equation (7), there is obtained

$$\rho \frac{Dh}{Dt} = \nabla \cdot (k\nabla T) - \kappa(4n^2 E_{bb} - \mathscr{E}) + \frac{Dp}{Dt} + \mu\Phi. \quad (8)$$

Note that the term $\kappa(4n^2 E_{bb} - \mathscr{E})$ appears with opposite signs in equations (3) and (8), indicating that a net gain of radiant energy occurs at the expense of molecular energy. When radiation terms are included in the energy equation the temperature is a more natural dependent variable than the enthalpy; hence the term containing enthalpy will usually be rewritten.

Since the radiant-energy flux vector $\mathbf{E}$ and the incident radiation $\mathscr{E}$ are both functions of temperature and the boundary conditions imposed on the intensity $I(\mathbf{r}, \mathbf{\Omega}, t)$, equations (7) and (8) are nonlinear integrodifferential equations. Unfortunately, in view of the existing known mathematical techniques available for the solution of integrodifferential equations, an exact solution of even a simplified energy equation does not seem possible.

† For the total or grey quantities the subscript λ is omitted.

Rosseland approximation

The grey-medium assumption (κ_λ = const.) reduced the complexity of the radiant-energy terms; however, the resulting energy equation is still rather formidable and further simplification is required. In the section below, an approximation for the radiant-flux vector is used; the assumptions made in deriving it are discussed in this section.

Rosseland [10] has shown that, for intense absorption and a system close to thermodynamic equilibrium, the radiant-energy-flux vector can be approximated by

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{3\kappa}\nabla(4n^2\sigma T^4) = -\frac{16n^2\sigma T^3}{3\kappa}\nabla T. \quad (9)$$

This relation also follows from the analog of simple kinetic-theory arguments. Photons travel with the velocity c. At any time, $\frac{1}{6}$ of the photons are moving in the x-direction and $\frac{1}{6}$ in the opposite direction. These photons execute a random walk through the absorption centers. If the definition of the Rosseland mean free path is used, the radiant energy flux is given by equation (9). Relation (9) can also be derived by expanding the equation for radiant flux in Taylor series [2, 11].

The Rosseland approximation leads to a considerable simplification in the expression for radiant flux. The integral representation is now replaced by a simple diffusion-type equation. The simplicity is, however, offset by its approximate nature and other disadvantages:

(1) The approximation is valid at points far (optically) from the bounding surface and only for intense absorption, i.e. when the fractional variation in temperature is small in a distance of one mean free path. The approximation breaks down in the vicinity of the surface, since it does not take into account radiation leaving from the surface; the presence of the surface changes the coefficient from 16/3 to 8/3.

(2) Equation (9) represents only the sum of the first two terms in the Taylor series expansion of the black-body emissive power E_{bb}, and therefore in some physical problems the error in truncating the series could be appreciable.

In spite of these shortcomings the Rosseland approximation has been used with success in a variety of problems ranging from transport of radiation through gases at low density to the study of the effects of radiation on blast waves caused by nuclear explosions. An extensive bibliography regarding Rosseland's approximation can be found in [9].

Boundary-layer equations

It is conceivable that under certain conditions radiation would cause an extremely thick thermal boundary layer. This raises the question whether or not boundary-layer approximations are valid when energy transport by radiation is significant. In the following analysis it is assumed that the boundary-layer-type energy equation is valid.

The co-ordinate system for the wedge is shown in Fig. 2. Making the usual Prandtl boundary

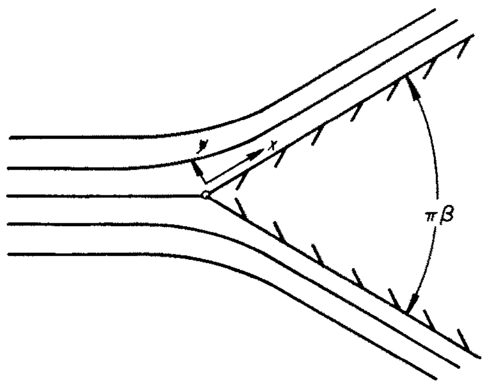

FIG. 2. Co-ordinate system for flow past a wedge.

layer assumptions [12] and utilizing the Rosseland approximation, the equations of conservation for steady state take the following form:

Conservation of mass:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0. \quad (10)$$

Conservation of momentum:

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right); \quad \frac{\partial p}{\partial y} = 0. \quad (11)$$

Conservation of energy:

$$\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(k_{\text{eff}} \frac{\partial T}{\partial y}\right) + u\frac{\partial p}{\partial x} + \mu\left(\frac{\partial u}{\partial y}\right)^2, \quad (12)$$

where

$$k_{\text{eff}} = k + k_r = k + \frac{16n^2\sigma T^3}{3\kappa} \quad (13)$$

is the effective thermal conductivity. The term $16n^2\sigma T^3/3\kappa$ can be considered as the "radiative conductivity".

The boundary conditions for this system of equations are assumed to be

$$\begin{aligned} u = v = 0, \quad & T = T_w \quad \text{at} \quad y = 0 \\ u = U, \quad & T = T_0 \quad \text{at} \quad y \to \infty. \end{aligned} \quad (14)$$

Similarity transformation

"Similar" solutions are discussed in [12]. They constitute a particularly simple class of solutions of $u(x, y)$ and cause the system of partial differential equations (10–12) to reduce to two ordinary differential equations.

It is proved in [12] that such similar solutions exist when the velocity of the potential flow is proportional to a power series of the length co-ordinate measured from the stagnation point, i.e. for

$$U = u_1 x^m. \quad (15)$$

To obtain the similar solution, the dimensionless co-ordinate η, first suggested by Faulkner and Skan [12], is introduced:

$$\eta = y\left[\frac{(m+1)}{2}\frac{U}{\nu^* x}\right]^{\frac{1}{2}}. \quad (16)$$

The asterisk designates a physical property evaluated at an arbitrary temperature T^*. The continuity equation (10) can then be integrated by introducing the stream function as

$$\psi(x, y) = \left[\frac{2}{(m+1)}\nu^* u_1 x^{m+1}\right]^{\frac{1}{2}} f(\eta). \quad (17)$$

The velocities in the conservation equations can be replaced through the definitions

$$\frac{\partial \psi}{\partial y} = \frac{\rho}{\rho^*} u \text{ and } \frac{\partial \psi}{\partial x} = -\frac{\rho}{\rho^*} v.$$

Thus the velocity components become

$$u = \frac{\rho^*}{\rho}\frac{\partial \psi}{\partial y} = \frac{\rho^*}{\rho} U f'(\eta) \quad (18)$$

and

$$v = -\frac{\rho^*}{\rho}\frac{\partial \psi}{\partial x} = -\frac{\rho^*}{\rho}\left[\frac{(m+1)}{2}\nu^* u_1 x^{m-1}\right]^{\frac{1}{2}} \times \left[f(\eta) + \left(\frac{m-1}{m+1}\right)\eta f'(\eta)\right]. \quad (19)$$

In the case that ρ, μ and k_{eff} are functions of temperature, the energy equation is nonlinear in temperature and the most convenient definition of dimensionless temperature θ is $\theta = T/T^*$. It is also possible to show that θ can be expressed as a function of η alone for the boundary conditions in this problem.

Using the Bernoulli equation to evaluate the pressure gradient and introducing the similarity variable η, the stream function ψ and the dimensionless temperature θ, the conservation equations (11) and (12) become

$$\left\{\left(\frac{\mu}{\mu^*}\right)\left[\left(\frac{\rho^*}{\rho}\right)' f' + \left(\frac{\rho^*}{\rho}\right) f''\right]\right\}' + f\left[\left(\frac{\rho^*}{\rho}\right)' f' + \left(\frac{\rho^*}{\rho}\right) f''\right] + \beta\left[\left(\frac{\rho}{\rho^*}\right) - \left(\frac{\rho^*}{\rho}\right) f'^2\right] = 0 \quad (20)$$

and

$$[(k_{\text{eff}}/k)\theta']' + N_{Pr} f \theta' = 0, \quad (21)$$

where the prime denotes differentiation with respect to η. In problems where energy transport by thermal radiation is appreciable in comparison to energy transport by conduction and convection, temperature levels will be high and the work of the pressure forces and viscous heat dissipation is unimportant and has therefore been neglected. The ratio k_{eff}/k is expressed as

$$k_{\text{eff}}/k = 1 + 4\theta^3/3N. \quad (22)$$

The dimensionless parameter N determines the relative role of conduction versus radiation term. It was discussed in a previous section that in the immediate vicinity of the wall the coefficient in equation (9) changes from 16/3 to 8/3. Therefore in the numerical calculations a constant of 2/3, instead of 4/3, was used in equation (22) for values of η smaller than 4 per cent of the thermal-boundary-layer thickness.

The boundary conditions for this system of differential equations are obtained from equations (14), (16) and (17) as

$$\left.\begin{array}{l} f = f' = 0, \quad \theta = \theta_w \text{ at } \eta = 0 \\ f' = 1, \quad \theta = \theta_0 \text{ at } \eta \to \infty. \end{array}\right\} \quad (23)$$

DISCUSSION OF RESULTS

Numerical integrations of equations (21) and (22) are reported for two distinct cases. In the first calculation, the effect of the parameter N is considered when the physical properties are assumed to be independent of temperature. Temperature distributions for a Prandtl number of 1 were calculated for both hot and cool walls. Since the energy equation is nonlinear and no suitable dimensionless temperature was found that would have eliminated the specification of particular values of θ_w and θ_0 in the boundary conditions (23), solutions have been obtained for several values of θ_w and θ_0.

In the second calculation, consideration was given to Pyrex glass, for which the viscosity is a very strong function of temperature. Both the velocity and temperature profiles are therefore affected by this dependence. Molten Pyrex glass was chosen because of interest in re-entry problem and because it is possible to change the absorption coefficient of the glass by addition of a carbonizing plastic [13]. The values of physical properties and the dependence of viscosity on temperature were taken from [14].

Temperature distributions

The velocity profiles for the constant physical property case are well known and are therefore not considered here. Temperature profiles have been obtained for fifty-four cases for various values of parameters of interest. For the sake of brevity only some of these are shown in Figs. 3 and 4 for a pressure gradient parameter $\beta = 0$, and cool and hot walls, respectively. Because of a programming error, some of the results reported in [9] for the case $\theta_w = \theta_0 = 0{\cdot}2$ are in error. When $\beta = \frac{1}{2}$, the results are similar in trend except that the temperature gradients are somewhat steeper. The dimensionless temperature varies monotonically across the boundary layer from the wall value to the free-stream value.

The physical nature of the results can be understood better when we note that the parameter N represents the relative role of energy transport by conduction to that of radiation. Three cases were considered: (i) $N = 10$, (ii) $N = 1{\cdot}0$ and (iii) $N = 0{\cdot}1$. In the first case conduction predominates, in the second case energy transport by molecular conduction is of the same order of magnitude as radiation and in the last case radiation predominates. As $N \to \infty$, the temperature profiles approach those for pure conduction. In all cases, the temperature profiles for values of $N = 10$ did not differ by more than about 2 per cent from those of pure conduction

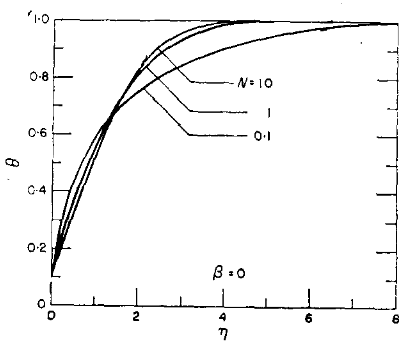

FIG. 3(a). Temperature profiles as functions of the similarity variable η for $N_{Pr} = 1{\cdot}0$, $\beta = 0$, $\theta_w = 0{\cdot}1$ and $\theta_0 = 1{\cdot}0$.

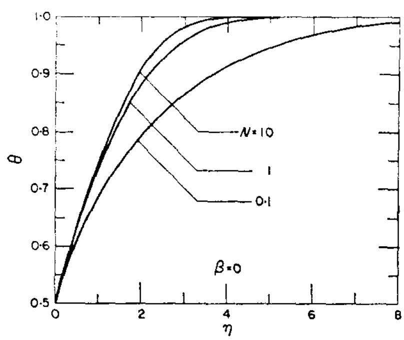

FIG. 3(b). Temperature profiles as functions of the similarity variable η for $N_{Pr} = 1{\cdot}0$, $\beta = 0$, $\theta_w = 0{\cdot}5$ and $\theta_0 = 1{\cdot}0$.

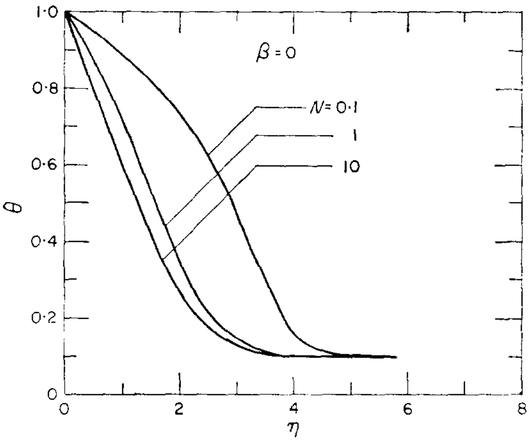

FIG. 4(a). Temperature profiles as functions of the similarity variable η for $N_{Pr} = 1{\cdot}0$, $\beta = 0$, $\theta_w = 1{\cdot}0$ and $\theta_0 = 0{\cdot}1$.

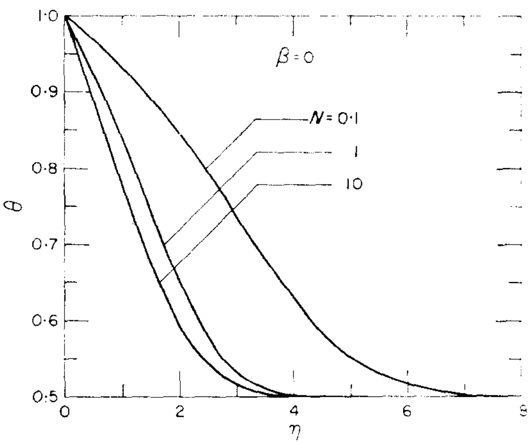

FIG. 4(b). Temperature profiles as functions of the similarity variable η for $N_{Pr} = 1{\cdot}0$, $\beta = 0$, $\theta_w = 1{\cdot}0$ and $\theta_0 = 0{\cdot}5$.

($N = \infty$) and therefore the curves for $N = \infty$ have not been included in the figures.

The study of temperature profiles obtained brings out the following conclusions:

(1) The effect of radiation is to thicken the thermal boundary layer as if to lower the Prandtl number.

(2) For a given value of N the departure of the temperature profile from that with $N = \infty$ is greater with the hot wall than with the cold wall.

Since Pyrex glass possesses no definite melting temperature, 2500°F was chosen as the boundary condition at the solid wall. The dimensionless stream function f, the velocity ratio f' and the shear function f'' distributions for Pyrex glass are plotted in Fig. 5 as functions of η. Because of the very strong temperature dependence of viscosity, the shear function is a maximum, not at the wall, but at some point away from it. The maximum shear occurs at different values of the similarity variable η for different κ values. The shear stress at the wall was found to vary little with the absorption coefficient. The temperatures are plotted in Fig. 6 for four values of the absorption coefficient, and it is seen from Figs. 5 and 6 that the thermal-boundary-layer thickness is about five times smaller than the momentum-boundary-layer thickness.

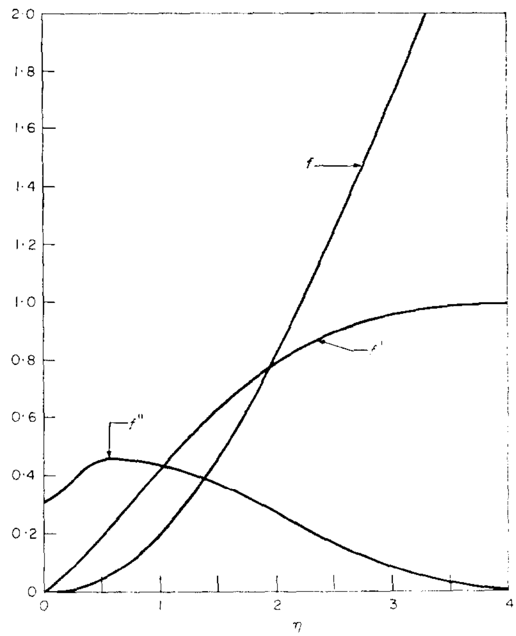

FIG. 5. Dimensionless stream function, velocity ratio and shear function vs. similarity variable η for Pyrex glass, $\beta = 0$, $\theta_w = 0{\cdot}664$, $\theta_0 = 1{\cdot}0$, $\kappa = 10^3$ ft^{-1} and $T^* = 4460$°R.

Heat transfer

As an illustration, the variation of heat transfer across the boundary layer is plotted in Fig. 7 in terms of the temperature gradient for the

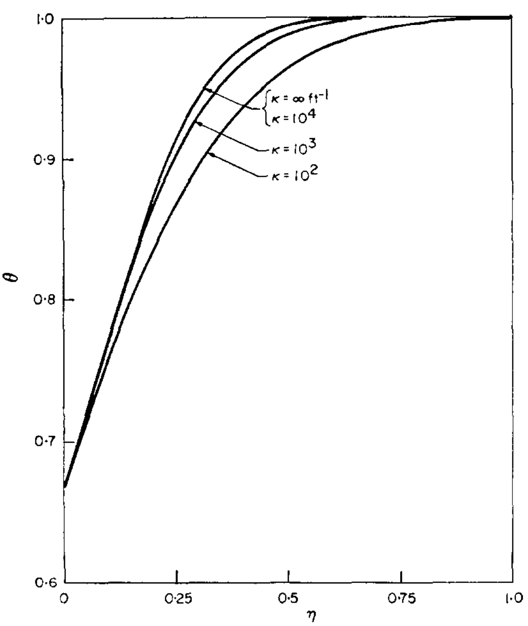

FIG. 6. Temperature profiles as functions of similarity variable η for Pyrex glass, $\beta = 0$, $\theta_w = 0{\cdot}664$, $\theta_0 = 1{\cdot}0$ and $T^* = 4460°$R.

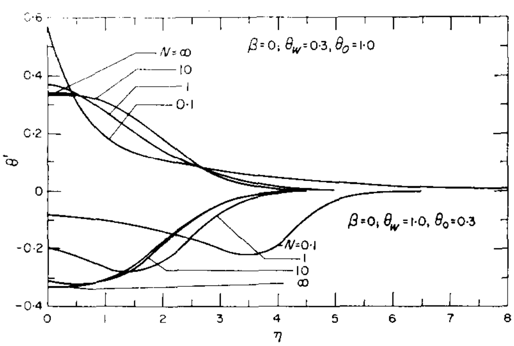

FIG. 7. Temperature gradients across the boundary layer.

pressure gradient parameter $\beta = 0$ and both cool as well as hot walls. Note that for the case of the cool wall, the temperature gradient is maximum at the wall, while for the hot wall the temperature gradient is maximum at some point away from the wall. The value of the similarity variable η at which the gradient is maximum increases with the decrease of N. These trends are due to the very nonlinear dependence of the radiant energy flux on the temperature and can readily be explained by considering the energy equation (21).

The zone adjacent to the surface where θ' is essentially constant for the case of large N and the cool wall decreases rapidly with the decrease in N. This is the zone where heat transfer is primarily by conduction, since the temperature level is still low and because of the nearly zero velocities in the neighborhood of the surface. For the case of the cool wall the local heat flux by conduction is maximum at the surface, but the radiant-energy flux by radiation, see equation (9), which is a function not only of the temperature gradient but also of the temperature, is maximum at some point in the boundary layer. On the other hand, for the hot wall the local radiant-energy flux is maximum at the surface, but the local conductive-energy flux is maximum at a point away from the surface.

The temperature gradients at the walls are shown plotted as functions of the temperature at the wall for the case of the cool wall in Figs. 8 and 9 and as the functions of the free-stream

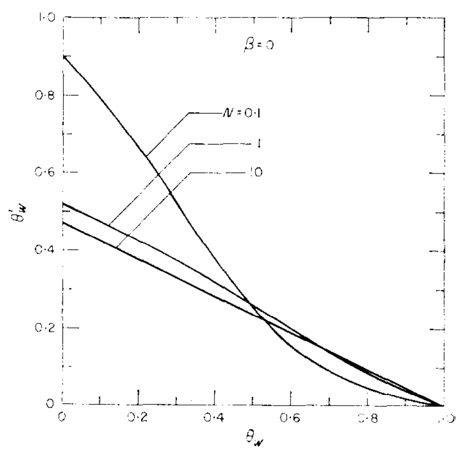

FIG. 8. Temperature gradients at the wall as functions of temperature at the wall for $\beta = 0$.

temperature for the case of the hot wall in Fig. 10. For a non-radiating medium ($N = \infty$) and the cool wall, the temperature gradients at the wall are less than 1 per cent smaller than for the case $N = 10$, and therefore separate curves were not drawn. For the purpose of comparison, the curve for the case $N = \infty$ has been included in Fig. 10. As expected θ_w' is practically linear with θ_w or θ_0 for $N = 10$ when energy transport by

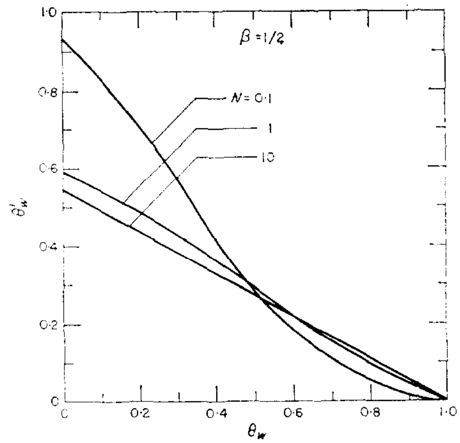

FIG. 9. Temperature gradients at the wall as functions of temperature at the wall for $\beta = \frac{1}{2}$.

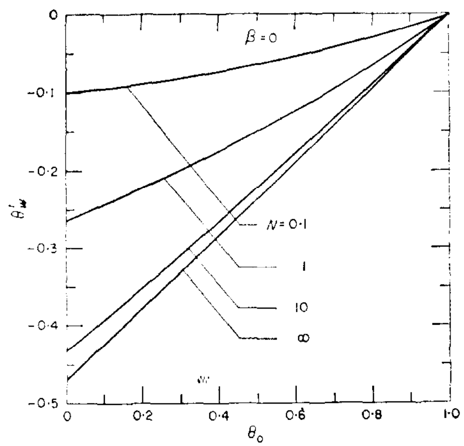

FIG. 10. Temperature gradients at the wall as functions of free stream temperature for $\beta = 0$.

conduction predominates. As the thermal radiation becomes the predominant mode of heat transfer, the temperature gradient at the wall departs more from linearity with decreasing N.

The total heat flux is the sum of conductive and radiative energy fluxes. For strongly absorbing medium and black surfaces the heat flux at the wall can be expressed as

$$q'' = q''_c + q''_r = -k\frac{\partial T}{\partial y}\bigg|_w - \frac{8n^2\sigma T^3}{3\kappa}\frac{\partial T}{\partial y}\bigg|_w. \quad (24)$$

It has been assumed that the Rosseland approximation (9) is also valid at the wall except that the coefficient was changed to 8/3 from 16/3. For non-black surface, the energy flux can be approximated by multiplying the second term on the right-hand side by the emissivity of the surface [2]. Since the distance y normal to the surface in the physical plane is related to the similarity variable η through equation (16), the total heat flux may be expressed as

$$\frac{q''x}{kT^*(N_{Re_x})^{\frac{1}{2}}} = -\left(\frac{m+1}{2}\right)^{\frac{1}{2}}\left(1 + \frac{2\theta_w^3}{3N}\right)\theta'_w. \quad (25)$$

The heat-transfer results calculated in this manner for $N_{Pr} = 1{\cdot}0$ are given in Table 1.

Table 1. *Some heat-transfer results for flow along a wedge in terms of the parameter* $q''x/kT^*\ (N_{Re_x})^{\frac{1}{2}}$

N	θ_w 0·1	0·3	0·5	0·7
	(a) *Cool wall*, $\beta = 0$			
10	−0·303	−0·237	−0·170	−0·102
1	−0·338	−0·272	−0·198	−0·122
0·1	−0·561	−0·435	−0·330	−0·221
	(b) *Cool wall*, $\beta = \frac{1}{2}$			
10	−0·401	−0·313	−0·224	−0·134
1	−0·444	−0·356	−0·258	−0·154
0·1	−0·694	−0·553	−0·420	−0·264
	θ_0			
	(c) *Hot wall*, $\beta = 0$			
10	0·296	0·232	0·165	0·100
1	0·286	0·231	0·173	0·102
0·1	0·501	0·432	0·348	0·233

It is seen from the results that the effect of radiation is to increase the heat-transfer rate. As expected, the radiant heat transfer increases with the increase in θ_w. The results of Table 1(c) show that there exists a minimum heat-transfer rate with N for low values of θ_0. The physical reasons for this trend are not apparent. The results also show that for a hundredfold variation in the parameter N the heat-transfer rate is increased only by about a factor of 2.

To provide a broader perspective for the effects of radiation on boundary-layer heat transfer it is of interest to compare the findings of Goulard and Goulard [15], who studied the other limiting case of heat transfer in a non-absorbing but emitting layer of gas, with those of this study. They found that heat transfer by convection to a cool wall ($\theta_w = 0{\cdot}572$) is reduced if the gas layer radiates; however, the additional radiative energy flux increases the total heat flux to the surface. This is in agreement with the results of this study. The temperature gradients at the walls, see Figs. 8 and 9, for approximate values of $\theta_w > 0{\cdot}5$ are lower than for those of a non-radiating medium, and therefore the convective heat-transfer rate is lower when energy transport by thermal radiation is present. The total heat transfer rate as given in Table 1(a) and (b) is higher, however, for a medium that radiates.

By way of generalization, it should be pointed out that the results obtained in this paper for an absorbing and emitting medium are also valid when scattering is also present. In the presence of an absorbing, emitting and scattering medium, the Rosseland approximation is modified [7, 9] by replacing the absorption coefficient κ in the denominator of equation (9) by the extinction coefficient γ, which is the sum of the absorption and scattering coefficients, and redefining the dimensionless parameter N as $N = k\gamma/4n^2\sigma T^{*3}$.

SUMMARY AND CONCLUSIONS

The transfer of energy in a boundary-layer flow of an incompressible and radiating medium has been studied. Because the conservation-of-energy equation is so complex when radiant energy fluxes are included, the solution of the boundary-layer equations is very cumbersome; for this reason, the Rosseland approximation for the radiant-energy-flux vector has been employed to simplify the energy equation. The approximation fails in the vicinity of the surface since it does not properly take into account radiation leaving from the surface, and therefore the temperature profiles near the boundary are in error.

The effect of radiation is to decrease θ_w' below those of a non-radiating medium for the case of the cool wall and approximate values of $\theta_w > 0{\cdot}5$ and to increase them for approximate values of $\theta_w < 0{\cdot}5$. On the other hand, for the hot wall the temperature gradients at the wall are lower than those for $N = \infty$ for all values of θ_0. The total heat-transfer rate, however, is always increased when the medium absorbs and emits thermal radiation.

The results reported in the paper are approximate because of the simplifications made, and in the future, refinements will have to be made to the analysis by more exact or integral-equation formulation of thermal radiation. It is expected, however, that the results of the approximate analysis will retain the significant quantitative aspects of the actual behavior at conditions studied here.

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Résumé—Cet article étudie les effets du rayonnement thermique sur la distribution de température et les échanges thermiques dans la couche limite autour d'un dièdre lorsque le fluide est absorbant ou émissif. Le problème général de la couche limite est posé sous forme d'une équation de transport pour le rayonnement thermique, d'une équation différentielle non linéaire pour la conservation de l'énergie et des équations habituelles pour la conservation de la masse et de la quantité de mouvement. Comme il n'existe pas de solution générale pour ces équations, on utilise l'approximation de Rosseland pour le flux de rayonnement afin de simplifier l'équation de l'énergie. Des solutions numériques sont données pour la distribution de température et la transmission de chaleur. Les calculs ont été faits pour un fluide dont le nombre de Prandtl est égal à l'unité aussi bien que pour du Pyrex fondu.

Zusammenfassung—Das Grenzschichtproblem einer strahlungabsorbierenden und emittierenden Flüssigkeit beim Umströmen eines Keils wird unter Berücksichtigung der Strahlungseinwirkung auf Temperaturverteilung und Wärmeübergang behandelt. Das gewöhnliche Grenzschichtproblem ist in Form einer Übergangsgleichung für thermische Strahlung formuliert, einer nicht linearen Integrodifferentialgleichung, die Energieerhaltung und die üblichen Gleichungen der Erhaltung von Masse und Impuls ausdrückt. Da keine allgemeine Lösung dieser Gleichungen üblich ist, wurde zur Vereinfachung der Energiegleichung die Rosseland-Näherung für den Strahlungswärmeflussvektor herangezogen. Numerische Lösungen für Temperaturverteilung und Wärmeübergang sind angegeben. Die Berechnungen erstrecken sich auf eine Flüssigkeit der Prandtl-Zahl 1 und auf geschmolzenes Pyrexglas.

Аннотация—Рассматривается проблема пограничного слоя для клина в излучающих и поглощающих средах при влиянии теплового излучения на распределение температуры и теплопередачу. Общая проблема пограничного слоя описывается дифференциальными уравнениями переноса, включая нелинейное интегродифференциальное уравнение для переноса энергии теплового излучения; они представляют собой уравнения сохранения энергии, сохранения массы и количества движения. Поскольку невозможно общее решение для этих уравнений, то используется апроксимация Росселанда для вектора лучистого потока тепла с тем, чтобы упростить уравнение энергии. Приводятся численные решения для распределения температуры и величины теплопереноса. Расчеты приведены для жидкости с числом $Pr = 1$, а также для расплавленного стекла Пирекс.